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COUNTERINJECTIVE MODULES AND DUALITY*

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We study the modules ${}_RU$ which are injective over their endomorphism ring and we apply the results obtained to faithfully balanced bimodules and rings with Morita duality. We also get necessary and sufficient conditions, in terms of the left R-module structure, for U to be an injective cogenerator over its endomorphism ring.

1. Introduction

It is well known that a faithfully balanced bimodule ${}_RU_T$ (i.e., such that $R=\mathrm{End}({}_TU)$ and $T=\mathrm{End}({}_RU)$) induces a Morita duality between R and T precisely when ${}_RU$ and U_T are injective cogenerators. An asymmetrical generalization of Morita duality has been given by Zelmanowitz and Jansen in [23] by considering '*duality R-modules*', i.e., bimodules ${}_RU_T$ such that ${}_RU$ is a finitely cogenerated linearly compact quasi-injective self-cogenerator and T is naturally isomorphic to $\mathrm{End}({}_RU)$. As it has been remarked in [23], these modules can be regarded as '1/2-Morita duality modules', because ${}_RU_T$ is a Morita duality module if and only if it is a duality R-module and a (right) duality S-module. However, if ${}_RU$ is a module and $T=\mathrm{End}({}_RU)$, then U_T can be an injective cogenerator without ${}_RU$ being finitely cogenerated nor linearly compact nor a self-cogenerator and thus the question arises of giving necessary and sufficient conditions on ${}_RU$ for U_T to be an injective cogenerator. This is done in Theorem 10, where the duality between subcategories of R-**Mod** and **Mod**-T induced by the contravariant functors Hom(–, U) is also used to characterize these modules.

A key result for this study is the characterization of the modules ${}_RU$ which are injective over its endomorphism ring which, following a well established use for properties over the endomorphism ring of a module, will be called *counterinjective* modules. This is done in Theorem 2 and the results obtained are applied to the symmetrical situation obtained by considering a faithfully balanced bimodule ${}_RU_T$ such that ${}_RU$ and U_T are injective. Unlike what happens with the condition of being a cogenerator, this is not enough for ${}_RU_T$ to be a Morita duality module but gives a

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Morita duality (in the sense of Colby and Fuller [3]) between the quotient categories of R-**Mod** and **Mod**-T modulo the torsion theories cogenerated by ${}_RU$ and U_T. Using this approach we show that R has a left Morita duality when, in addition to requiring that ${}_RU$ and U_T are injective modules with ${}_RU_T$ faithfully balanced, we assume that the class of modules of U-dominant dimension ≥ 2 is closed under direct unions. If some additional finiteness conditions are added, then we get results which extend to faithfully balanced bimodules some of the usual characterizations of quasi-Frobenius rings. For instance, a module ${}_RU$ is Σ-injective (i.e., every direct sum of copies of ${}_RU$ is injective) if and only if the ACC holds on the set $A_R(U)$ of left ideals of R that are annihilators of subsets of U. Then, ${}_RU$ and U_T become artinian cogenerators, assuming Σ-injectivity of ${}_RU$ and U_T. If we start with a quasi-injective module and assume the dual condition of Δ-quasi-injectivity (i.e., $A_R(U)$ satisfies the DCC which by the Teply–Miller theorem implies the ACC), with $T=\mathrm{End}({}_RU)$ and $B=\mathrm{Biend}({}_RU)$, then T is a right artinian ring with a right Morita duality with B induced by ${}_BU_T$ if and only if ${}_RU$ has the DCC on U-closed submodules and ${}_RU$ cogenerates all the cokernels of homomorphisms of the form $U \to U^n$. Furthermore, if the ring R is commutative, then the DCC on closed submodules can be replaced by the condition of ${}_RU$ being finite-dimensional. In view of the fact that a noetherian Δ-quasi-injective module always has DCC on closed submodules, this gives some information on the structure of noetherian Δ-quasi-injective modules which had been posed by Faith in [6]. If the cogenerating condition of ${}_RU$ expressed above does not hold, then ${}_BU_T$ is not necessarily a Morita duality module but T still has a right Morita duality.

2. Definitions and notation

Throughout this paper R denotes an associative ring with identity and R-**Mod** the category of left R-modules. If U is a module, then we will say that a module X is (finitely) U-generated (resp. U-cogenerated) if it is a quotient (resp. a submodule) of a (finite) direct sum (resp. direct product) of copies of U. We recall that a module U is quasi-injective if, for every submodule X of U, the canonical homomorphism $\mathrm{Hom}_R(U,U) \to \mathrm{Hom}_R(X,U)$ is an epimorphism. We also recall that a module M is called *FP-injective* (resp. *semi-injective*) if $\mathrm{Ext}(F,M)=0$ for every finitely presented (resp. finitely presented cyclic) module F.

An R-module X is called a Δ-module when R satisfies the descending chain condition (DCC) on annihilators of subsets of X. If X is, furthermore, injective or quasi-injective, then X is said to be a Δ-injective or a Δ-quasi-injective module. The ascending chain condition shall be abbreviated by ACC.

If $T=\mathrm{End}({}_RU)$, then ${}_RU_T$ is a bimodule and the U-dual functors $\mathrm{Hom}_R(-,U)$ and $\mathrm{Hom}_T(-,U)$ will be denoted, as usual, by $(\ )^*$. The evaluation maps yield natural transformations $\phi : 1_{R\text{-}\mathbf{Mod}} \to (\ )^{**}$ and $\phi : 1_{\mathbf{Mod}\text{-}T} \to (\ )^{**}$. A module X (in R-**Mod** or in **Mod**-T) is U-cogenerated precisely when ϕ_X is a monomorphism and if ϕ_X is an isomorphism, X is said to be U-reflexive. X is called U-dense when for each $h \in X^{**}$ and each finite family $f_1, \dots, f_n \in X^*$, there exists an $x \in X$ such that $h(f_i) = f_i(x)$ for $i = 1, \dots, n$ [21, p. 431].

If X and U are R-modules, a submodule Z of X will be called (finitely) U-closed when X/Z is (finitely) U-cogenerated. This means that Z is closed in the Galois connection between submodules of X and submodules of $\mathrm{Hom}_R(X, U)$ defined by the usual annihilator mappings [1, Proposition 3.5]. Thus if we denote, as usual, for submodules Z of X and L of X^*, $r_{X^*}(Z) = \{f \in X^* \mid f(z) = 0 \text{ for each } z \in Z\}$ and $\ell_X(L) = \{x \in X \mid f(x) = 0 \text{ for each } f \in L\}$, then Z is U-closed precisely when $Z = \ell_X r_{X^*}(Z)$ $(= \ell r(Z)$, for short). Similarly, L is U-closed in X^* when $L = r_{X^*}\ell_X(L) = r\ell(L)$. An important role in our development will be played by the following class of modules:

Definition. Let $_RU$ be a left R-module and $T = \mathrm{End}(_RU)$. $_RU$ will be called *counterinjecive* when U_T is an injective module.

We refere the reader to [2] and [21] for all the ring and module-theoretic notions used in the text.

3. Counterinjective modules

Let $_RU$ be a left R-module. By [22, Statz 1.8] and [5, Theorem 4], if $T = \mathrm{End}(_RU)$, then U_T is FP-injective (resp. semi-injective) if and only if $_RU$ cogenerates all cokernels of morphisms $U^m \to U^n$ (with $m = 1$). We want to give a characterization of when $_RU$ is counterinjective. For this, if X is a left R-module we will say that X is *U-linearly compact* when each finitely solvable system of congruences of the form $x \equiv x_i \pmod{X_i}$, with the X_i U-closed submodules of X, is solvable. Of course, if every submodule of X is U-closed, this concept reduces to that of a linearly compact module (in the discrete topology). As in [21, 29.7], the U-linearly compact modules can be characterized in the following way:

Proposition 1. *Let $_RX$ and $_RU$ be left R-modules. Then the following conditions are equivalent:*

(i) *X is U-linearly compact.*

(ii) *Each finitely solvable system of congruences $x \equiv x_i \pmod{X_i}$, where the X_i are finitely U-closed submodules of U is solvable.*

(iii) *If $\{X_i\}_I$ is an inverse system of (finitely) U-closed submodules of X, then $\varprojlim\{X/X_i\}$ is canonically isomorphic to $X/\bigcap X_i$.*

Proof. (i) $\Rightarrow$ (iii). If $\{X_i\}$ is an inverse system of U-closed submodules of X, one gets an exact sequence of inverse systems:

$$0 \longrightarrow X_i \xrightarrow{j_i} X \xrightarrow{p_i} X/X_i \longrightarrow 0$$

and, taking inverse limits, an exact sequence in R-**Mod**:

$$0 \longrightarrow \bigcap X_i \longrightarrow X \xrightarrow{\varprojlim p_i} \varprojlim X/X_i$$

An element of $\varprojlim X/X_i$ is of the form $(x_i + X_i) \in \prod X/X_i$, satisfying the condition that $x_j - x_i \in X_i$ for $X_j \subseteq X_i$. This element defines a system of congruences $x \equiv x_i$ $(\text{mod } X_i)$ which is, obviously, finitely solvable and hence solvable by hypothesis. Then, if z is a solution, we have that $(x_i + X_i) = (\varprojlim p_i)(z)$ and so $\varprojlim p_i$ is an epimorphism, proving (iii).

(iii) $\Rightarrow$ (ii) Assume that (iii) holds for finitely closed submodules of X. Notice also that to prove (ii) we may assume, replacing the modules X_i corresponding to the system $x \equiv x_i$ (mod X_i) by their finite intersections, that we have a finitely solvable system $x \equiv x_i$ (mod X_i) where the $\{X_i\}_I$ form an inverse system of finitely closed submodules of X. We thus have an exact sequence $0 \to \bigcap X_i \to X \to \varprojlim X/X_i \to 0$. Then $(x_i + X_i)_i$ is clearly an element of $\varprojlim X/X_i$ and from the exactness of the above sequence it follows that the system $x \equiv x_i$ (mod X_i) is solvable.

(ii) $\Rightarrow$ (i) Let $x \equiv x_i$ (mod X_i) be a finitely solvable system with each X_i closed. Then $X_i = \ell\imath(X_i)$, with each $\imath(X_i)$ a right T-submodule of $X^* = \text{Hom}_R(X, U)$. Let $\imath(X_i) = \sum Z_{i_k}$, where, for each $i \in I$, Z_{i_k} ranges over the set of finitely generated T-submodules of $\imath(X_i)$ and the sum is directed. Then, $X_i = \ell(\sum Z_{i_k}) = \bigcap \ell(Z_{i_k})$, where $\{\ell(Z_{i_k})\}_k$ is an inverse system of finitely closed submodules of X, for each $i \in I$. Calling $X_{i_k} = \ell(Z_{i_k})$ and considering the system of congruences $x \equiv x_{i_k}$ (mod X_{i_k}), where we set $x_i = x_{i_k}$ for each index i_k, we see that this system is finitely solvable and hence solvable by (ii). This clearly implies that the original system $x \equiv x_i$ (mod X_i) is solvable and thus the proof is complete. □

We can now give a criterion for the counterinjectivity of ${}_RU$.

Theorem 2. *A left R-module ${}_RU$ is counterinjective if and only if the following conditions hold*:

(i) *Every cokernel of a homomorphism of the form $U^m \to U^n$ (or $U \to U^n$) is U-cogenerated, equivalently U_T is FP-injective.*

(ii) *U is U-linearly compact.*

Proof. As it has been already remarked, we know from results of Würfel and Damiano that U_T is FP-injective (resp. semi-injective) if and only if condition (i) (resp. condition (i) for $m = 1$) holds. Thus we have to show that, in the presence of this condition, U_T is injective if and only if (ii) is satisfied.

Assume first that U_T is injective and, using Proposition 2, consider an inverse system $\{U \xrightarrow{p_i} U_i\}_I$ of U-cogenerated quotients of U. Denoting, as usual, by $*$ the U-dual functors, we have, as a consequence of the injectivity of U_T, commutative diagrams with exact rows,

$$\begin{array}{ccccc} U & \xrightarrow{p_i} & U_i & \longrightarrow & 0 \\ {\scriptstyle \phi_U}\downarrow & & \downarrow{\scriptstyle \phi_{U_i}} & & \\ U^{**} & \xrightarrow{p_i^{**}} & U_i^{**} & \longrightarrow & 0 \end{array}$$

ϕ_U is an isomorphism and, since each U_i is U-torsionless, ϕ_{U_i} is a monomorphism for each $i \in I$. Therefore we see that ϕ_{U_i} is an isomorphism (i.e., each U_i is U-reflexive), so that we can identify p_i^{**} with p_i. Then, using the facts that the p_i^* form a direct system of monomorphisms (corresponding to right ideals of T) and the functor $\mathrm{Hom}_R(-, U)$ takes direct limits into inverse limits, we get that $\varprojlim p_i = \varprojlim p_i^{**} = (\varinjlim p_i^*)^*$. Since the direct limit functor is exact we get that $\varinjlim p_i^*$ is a monomorphism and hence $\varprojlim p_i$ is an epimorphism, which proves (ii).

Conversely, if conditions (i) and (ii) hold, we already know that U_T is FP-injective (or semi-injective). Let then J be a right ideal of T (with inclusion $j : J \to T$) and $\{J_i\}_I$ the direct system of all the finitely generated right ideals contained in J, with canonical inclusions $j_i : J_i \to T$. From the fact that U_T is semi-injective it follows that the canonical homomorphisms $U \to J_i^*$ are epimorphisms. Thus we have an inverse system $\{U \to J_i^*\}_I$ where the J_i^* are U-cogenerated quotients of U. Using (ii) and Proposition 1 we conclude that $\varprojlim j_i^* : U \to \varprojlim J_i^*$ is also an epimorphism. But then we have that $\varprojlim j_i^* = (\varinjlim j_i)^* = j^*$ and hence U_T is injective. □

Remarks. In [10] a module U over a commutative ring is called *semi-compact* when each finitely solvable system $x \equiv x_i \pmod{U_i}$ is solvable, whenever the submodules U_i are annihilators of ideals of R. Since these annihilators are obviously U-closed submodules of U, we see that if U is U-linearly compact, then it is also semi-compact and hence the fact that any injective module over a commutative ring is semi-compact [10, Proposition 2] is contained in Theorem 2.

Using Theorem 2 one can recover as easy corollaries a few results scattered in the literature. We mention, among them, [4, Theorem 2.8] which asserts that if ${}_RU$ is an injective artinian module which cogenerates an exact torsion theory of R-**Mod**, then ${}_RU$ is counterinjective. In fact, as we will see later on, U_T is, actually, an injective cogenerator in this case. Other straightforward consequences of Theorem 2 are [13, Corollary 1, p. 119; 18, Corollary 2, p. 342; 9, Proposition 2] and the last equivalence of [14, Theorem]. Also, it is easy to check that if $U = \bigoplus U_i$ is an infinite direct sum of nonzero modules, then U is not U-linearly compact and hence it cannot be counterinjective [17, Theorem 1].

It is easy to give examples of counterinjective modules which are not linearly compact (the most obvious one is ${}_{\mathbb{Z}}\mathbb{Q}$!). Even a faithfully balanced injective and counterinjective module need not be linearly compact. For instance, a Von Neumann regular ring R is always (left and right) self-FP-injective and hence it follows from Theorem 2 that it is right self-injective if and only if ${}_RR$ is R-linearly compact. However, a nonsemisimple left and right self-injective regular ring is not left nor right linearly compact.

Corollary 3. *Let $_RU_T$ be a faithfully balanced bimodule such that $_RU$ is a cogenerator and U_T is injective. Then R has a left Morita duality.*

Proof. From Theorem 2 it follows that $_RU$ is a linearly compact cogenerator and thus it is clear that the minimal cogenerator of R-**Mod** is also linearly compact. On the other hand, R is left linearly compact by [13, Corollary 2, p. 119] and so it has a left Morita duality by [12]. □

Next, we study the duality defined by a counterinjective module. Motivated by Proposition 2, we will say that an object X of a Grothendieck category $\mathscr{C}$ is linearly compact when, for each inverse system of epimorphisms $\{p_i : X \to X_i\}_I$ in $\mathscr{C}$, the canonical morphism $\varprojlim p_i : X \to \varprojlim X_i$ is also an epimorphism. This concept will be applied in the case of the quotient category $\mathscr{C}_U$ of R-**Mod** modulo the (hereditary) torsion theory cogenerated by an injective module $_RU$ (see [19] for the definition). $\mathscr{C}_U$ can be identified with the full subcategory of R-**Mod** whose objects are all the modules X such that U-dom dim $X \geq 2$, i.e., such that there exists an exact sequence $0 \to X \to X_1 \to X_2$, in which X_1 and X_2 are direct products of copies of $_RU$.

Theorem 4. *Let $_RU$ be a module and $T = \mathrm{End}(_RU)$. Then the following statements hold*:

(i) *If X is a U-cogenerated U-linearly compact left R-module such that the cokernel of each homomorphism $X \to U^n$ is U-cogenerated, then X is U-reflexive. If, furthermore, $_RU$ is counterinjective, then the converse holds.*

(ii) *If $_RU$ is injective and counterinjective, then the U-reflexive R-modules are precisely the U-linearly compact modules X of $\mathscr{C}_U$ such that each epimorphism $X \to Y$ in $\mathscr{C}_U$ is an epimorphism in R-**Mod**. In particular, they are linearly compact objects of $\mathscr{C}_U$. In this case, the class of U-reflexive modules is closed under subobjects and quotient objects in $\mathscr{C}_U$.*

Proof. (i) Let X be a U-linearly compact U-cogenerated module such that each homomorphism of the form $X \to U^n$ has U-cogenerated cokernel. In order to show that $_RX$ is U-reflexive, it is enough to prove that the canonical homomorphism $\phi_X : X \to X^{**}$ is an epimorphism. Arguing as in the proof of [21, 47.8], we see that if $\{L_i\}_I$ is the family of all the finitely generated right T-submodules of X^*, then $\{r\ell(L_i)\}_I$ is also a direct system of submodules of X^* and $X^* \cong \varinjlim r\ell(L_i)$. Furthermore, $r\ell(L_i) \cong \mathrm{Hom}_R(X/\ell(L_i), U) = (X/\ell(L_i))^*$ and so, if we consider the inverse system of homomorphisms $X^{**} \to (X/\ell(L_i))^{**}$ we get that $X^{**} \cong (\varinjlim (X/\ell(L_i))^*)^* \cong \varprojlim (X/\ell(L_i))^{**}$. On the other hand, each $X/\ell(L_i)$ is U-dense by [21, 47.7] and since it is, by definition, finitely U-cogenerated, it follows from [21, 47.6] that $X/\ell(L_i)$ is, in fact, U-reflexive. Now, the proof can be completed as in [21, 47.8], for, using the fact that X is U-linearly compact, we get a commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & \varprojlim X/\ell(L_i) \\ {\scriptstyle \phi_X}\downarrow & & \downarrow{\scriptstyle q}\ \cong \\ X^{**} & \xrightarrow[\ p\ ]{\cong} & \varprojlim (X/\ell(L_i))^{**} \end{array}$$

in which g is an epimorphism and p and q are isomorphisms, so that ϕ_X is also an epimorphism.

Conversely, assume that ${}_RU$ is counterinjective and X is U-reflexive. Since U^n is obviously U-reflexive, we see that the cokernel of each homomorphism $X \to U^n$ is U-reflexive and hence U-cogenerated. Moreover, if $\{p_i : X \to X_i\}_I$ is an inverse system with the X_i U-cogenerated quotients of X, then we get a direct system in **Mod**-T $\{p_i^* : X_i^* \to X^*\}_I$ which gives a monomorphism $\varinjlim p_i^* : \varinjlim X_i^* \to X^*$. Taking U-duals and bearing in mind that the U_i are U-reflexive, we get an epimorphism

$$(\varinjlim p_i^*)^* \cong \varprojlim p_i^{**} : X^{**} \cong X \to \varprojlim X_i^{**} \cong \varprojlim X_i .$$

Thus we may identify the epimorphism $(\varinjlim p_i^*)^*$ with $\varprojlim p_i$ and so we see that X is U-linearly compact.

(ii) First, observe that the image of the functor $\mathrm{Hom}_T(-, U)$ is contained in $\mathscr{C}_U$ and hence it follows from (i) that if X is U-reflexive, then X is a U-linearly compact module of $\mathscr{C}_U$. Now, if $p : X \to Y$ is an epimorphism in $\mathscr{C}_U$, then $\mathrm{Im}\, p$ is also reflexive and hence belongs to $\mathscr{C}_U$, so that p is, in fact, an epimorphism of R-**Mod**. The converse is also clear, for if X is a U-linearly compact module of $\mathscr{C}_U$ such that each epimorphism $X \to Y$ in $\mathscr{C}_U$ is an epimorphism in R-**Mod**, then the cokernel of $f : X \to U^n$ in $\mathscr{C}_U$ coincides with the cokernel in R-**Mod** and is U-cogenerated, so that X is U-reflexive by (i).

On the other hand, if X is a U-reflexive R-module and $\{p_i : \mathrm{X} \to X_i\}_I$ an inverse system of epimorphisms in $\mathscr{C}_U$, then we see that, in fact, these are epimorphisms in R-**Mod** and the X_i are U-cogenerated. Since the inclusion functor of $\mathscr{C}_U$ in R-**Mod** has a left adjoint, it preserves inverse limits and hence it is clear that X is a linearly compact object of $\mathscr{C}_U$.

Finally, observe that the subobjects in $\mathscr{C}_U$ of a reflexive module X are precisely the U-closed submodules of X, which are obviously U-reflexive. On the other hand, we have seen that the quotients of X in $\mathscr{C}_U$ are U-cogenerated quotients of X in R-**Mod**, so that they are also U-reflexive. □

Remarks. Observe that, in the hypotheses of Theorem 4(ii), i.e., for an injective and counterinjective module ${}_RU$, if it is assumed that ${}_RU_T$ is, furthermore, a faithfully balanced bimodule, then the situation is made completely symmetric. In this case, if we let $\mathcal{D}_U$ be the quotient category of **Mod**-T modulo the torsion theory cogenerated by U_T, we see that R and S are reflexive generators of $\mathcal{C}_U$ and $\mathcal{D}_U$ respectively, and so $\mathrm{Hom}_R(-, U)$ and $\mathrm{Hom}_T(-, U)$ define a Morita duality between $\mathcal{C}_U$ and $\mathcal{D}_U$ in the sense of [3]. As in the case of Morita dualities for categories of modules, the reflexive objects are linearly compact in $\mathcal{C}_U$ and $\mathcal{D}_U$. If, furthermore, $\mathcal{C}_U$ and $\mathcal{D}_U$ are exact subcategories of R-**Mod** and **Mod**-T, then it is clear that the converse also holds. An example of this situation is obtained by considering a (Von Neumann) regular left and right self-injective ring and taking ${}_RU_T = {}_RR_R$. The associated quotient categories are, in this case, the categories of nonsingular injective modules which, as it is well known, are exact subcategories of R-**Mod** and **Mod**-R [19, Proposition IX.2.12]. Therefore, the dual functors $\mathrm{Hom}_R(-, R)$ induce a duality between the subcategories of linearly compact objects of these categories. Observe that, unlike the case of Morita dualities, R need not be semiperfect (nor finite-dimensional). However, if we add some additional conditions, the situation improves and we have:

Theorem 5. *Let ${}_RU$ be an injective and counterinjective module. If every direct sum of copies of ${}_RU$ has U-*dom dim ≥ 2, *then T is semiperfect. If U is faithfully balanced and the class of modules of U-*dom dim ≥ 2 *is closed under direct unions, then R has a left Morita duality.*

Proof. In order to show that T is semiperfect, we have to prove that ${}_RU$ is a finite-dimensional module (see, e.g. [19, Proposition XIV.1.7]). Let $\{Y_i\}$ be an independent family of submodules of ${}_RU$. Then, replacing each Y_i by its U-closure if necessary, we may assume that the Y_i are, in fact, closed submodules of U. Choose $x_i \in Y_i$ arbitrary for each $i \in I$ and set $X_i = \bigoplus_{j \neq i} Y_j$. Then, using the facts that U-dom $\dim(Y_i) \geq 2$ for each $i \in I$ (because each Y_i is a closed submodule of ${}_RU$) and U-dom $\dim(U^{(K)}) \geq 2$ for each set K (by hypothesis), it is not difficult to see that U-dom $\dim(X_i) \geq 2$ for each $i \in I$ (this is, essentially, a part of [19, Exercise XII.1.1]). Thus it is clear that each X_i is also a U-closed submodule of U. Since by Theorem 2 ${}_RU$ is rationally linearly compact and, clearly, the system of congruences $x \equiv x_i \pmod{X_i}$ is finitely solvable, we have that it is solvable. Therefore, the system has a solution in $\bigoplus Y_i$ and this implies that there can be only a finite number of nonzero x_i, so that ${}_RU$ is finite-dimensional.

Assume now that $R = \mathrm{End}(U_T)$ and the class of R-modules of U-dom dim ≥ 2 is closed under direct unions (of submodules of any given module). Let J be a left ideal of R and write $J = \sum J_i$, where the J_i are the finitely generated left ideals contained in J and the union is directed. It is easily checked that, since R is U-reflexive by hypothesis, each finitely generated left ideal of R is U-reflexive and so we have that U-dom $\dim(J_i) \geq 2$. Thus it follows from our hypotheses that U-dom $\dim(J) \geq 2$. But, since U-dom $\dim(R) \geq 2$, this clearly implies that J is a U-closed left ideal of R, so that each cyclic left R-module is cogenerated by ${}_RU$ and hence ${}_RU$ is a cogenerator. As U_T is injective, it follows from Corollary 3 that R has a left Morita duality. □

From the proof of the above theorem we get the following characterization of Morita duality bimodules: A bimodule ${}_RU_T$ is a Morita duality module if and only if it is faithfully balanced, injective and counterinjective, and the classes of modules of U-dom dim ≥ 2 (in R-**Mod** and in **Mod**-T) are closed under direct unions. The following result extends [6, Corollary 10.14]:

Corollary 6. *Let ${}_RU_T$ be a faithfully balanced bimodule such that ${}_RU$ is Σ-injective (resp. Δ-injective) and U_T is injective. Then R is a left noetherian (resp. artinian) ring with a left Morita duality.*

Proof. As it is well known, ${}_RU$ Σ-injective is equivalent to the ACC on U-closed left ideals of R. By [19, Proposition XIII.2.4 and XIII.1.2], this implies that the class of modules of U-dom dim ≥ 2 is closed under direct unions and so we have by the proof of the above theorem that ${}_RU$ is a linearly compact cogenerator and R has a left Morita duality. Furthermore, since ${}_RU$ is a cogenerator, it is also clear that R is actually left noetherian. The proof of the parenthetical case proceeds in a similar way using the fact that by the Teply–Miller theorem [6, Theorem 7.1] any Δ-injective module is Σ-injective. □

If we strengthen a little bit the hypotheses of the above corollary by making them symmetric, we get the following characterization of faithfully balanced Σ-injective bimodules:

Corollary 7. *Let ${}_RU_T$ be a faithfully balanced bimodule. Then the following conditions are equivalent:*

(i) *${}_RU$ and U_T are Σ-injective.*

(ii) *${}_RU$ and U_T are artinian injective cogenerators.*

In this case ${}_RU_T$ induces a Morita duality between the left noetherian ring R and the right noetherian ring T.

Proof. By Corollary 6 and its proof, if (i) holds, ${}_RU$ and U_T are linearly compact cogenerators and R (resp. T) is left (right) noetherian. Since there is an order-inverting bijective correspondence between (U-closed) submodules of ${}_RU$ (U_T) and (U-closed) right (left) ideals of T (R) [1, Proposition 4.3] we see that ${}_RU$ and U_T are artinian. The converse and the last statement are clear. □

The ring R in the above corollary need not be left artinian (e.g., if R is a non-artinian commutative local noetherian complete ring and U the minimal co-generator) but if we strengthen our conditions still a little bit more, we get

Corollary 8. *Let ${}_RU_T$ be a faithfully balanced bimodule. The following conditions are equivalent:*

(i) ${}_RU$ *is* Δ*-injective and* U_T *is* Σ*-injective.*

(ii) ${}_RU$ *and* U_T *are injective cogenerators of finite length.*

In this case R is left artinian, T is right artinian and ${}_RU_T$ *induces a Morita duality between R and T.*

Proof. By Corollary 7, ${}_RU$ and U_T are artinian injective cogenerators. Furthermore, since ${}_RU$ is Δ-injective, U_T is noetherian [6, Corollary 5.4] and hence of finite length. This in turn implies that $R=\text{End}(U_T)$ is left artinian and hence that ${}_RU$ has finite length, so that the result is clear. □

4. Quasi-injective modules

In the remainder of the paper we will focus on the dualities induced by quasi-injective modules (see [11] for a comprehensive study of the topological dualities associated with a quasi-injective self-cogenerator).

We are now going to consider a situation which is more general than that of Corollary 8 but where we can still ensure that T is a right artinian ring with a right Morita duality.

Theorem 9. *Let* ${}_RU$ *be a quasi-injective module and* $T=\text{End}({}_RU)$. *Then the following conditions are equivalent:*

(i) ${}_RU$ *is a* Δ*-module with DCC on U-closed submodules.*

(ii) ${}_RU$ *is a* Δ*-module and has a finitely generated submodule X such that* $\text{Ann}_T(X)=0$.

(iii) ${}_RU$ *has the DCC on U-closed submodules and* U_T *is finitely generated.*

(iv) *T is right noetherian and* U_T *is finitely generated.*

(v) *T is a right artinian ring with a right Morita duality and* U_T *is a finitely generated cogenerator.*

If these conditions hold, T has a right Morita duality with $B=\text{Biend}({}_RU)$ *induced by* ${}_BU_T$ *if and only if, either* U_T *is quasi-injective or* ${}_RU$ *cogenerates all the cokernels of homomorphisms* $U\to U^n$. *On the other hand, if R is commutative, then conditions* (i)–(v) *are equivalent to* ${}_RU$ *being a finite-dimensional* Δ*-quasi-injective module.*

Proof. (i) ⇒ (ii) Let $\bar{R}=R/\text{Ann}_R(U)$. By [6, Corollary 5.6A], ${}_{\bar{R}}U$ is an injective module and using the Teply-Miller theorem (see [1, Theorem 7.11]), we see that ${}_{\bar{R}}U$ (and hence ${}_RU$) has the ACC on U-closed submodules. By [1, Proposition 6.1], this implies that ${}_{\bar{R}}U$ has a finitely generated $\bar{R}$-submodule X such that $\text{Hom}_{\bar{R}}(U/X,U)=0$. This clearly implies that X is a finitely generated R-submodule of U such that $\text{Ann}_T(X)=0$.

(ii) ⇒ (iii) Since R has the DCC on U-closed left ideals, U_T is noetherian (see, e.g. [1, Corollary 4.3]). On the other hand, since there exists a finitely generated $\bar{R}$-submodule X of U such that $\mathrm{Hom}_{\bar{R}}(U/X, U) = 0$, it follows from [1, Corollary 6.4] that ${}_RU$ has the DCC on U-closed submodules.

(iii) $\Rightarrow$ (iv) This follows from [1, Corollary 4.3].

(iv) $\Rightarrow$ (v) By [6, Corollary 7.5], T has nilpotent radical. Since T is semiperfect (for ${}_RU$ is finite-dimensional) we have that T is right artinian (see [8, Corollary 1.5]). Now, each finitely generated right T-module Z is finitely presented and hence U-reflexive, so that, in particular, it is U-cogenerated. This implies that if C is a simple right T-module and $M \subseteq E(C)$ a nonzero finitely generated submodule of its injective hull, then M embeds in U_T. Since U_T has finite length we see that $E(C)$ has ACC on finitely generated submodules and hence it is a noetherian right T-module. Therefore $E(C)$ embeds in U_T and so U_T is a finitely generated cogenerator. Furthermore, it is clear that T has a right Morita duality.

(v) $\Rightarrow$ (i) This is again a straightforward consequence of [1, Corollary 4.3].

Observe now that T has a right Morita duality with B, induced by ${}_BU_T$, if and only if U_T is, furthermore, an injective module. Since T embeds in a finite direct product of copies of U_T, this is, in turn, equivalent to U_T being quasi-injective. On the other hand, since T is right artinian, U_T is injective if and only if it is FP-injective, i.e., if and only if ${}_RU$ cogenerates the cokernels of homomorphisms of the form $U \to U^n$.

Finally, we examine the case in which R is commutative. First, it is clear that if ${}_RU$ satisfies the equivalent conditions (i)–(v), then ${}_RU$ is finite-dimensional, so that all that remains to be proved is that if ${}_RU$ is a finite-dimensional Δ-quasi-injective module, then ${}_RU$ has the DCC on U-closed submodules. Observe that, by replacing R by $\bar{R}$ if necessary, we may assume that ${}_RU$ is a finite-dimensional Δ-injective module. By [1, Theorem 11.34], such a module has a decomposition $U = \bigoplus U_i^{r_i}$, with $r_i \geq 1$ and such that $U_i \cong E(R/\mathfrak{p}_i)$, where $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ are the primes associated to U and $R_{\mathfrak{p}_i}$ is an artinian ring for each $i = 1, \ldots, n$. Using [1, Proposition 11.8], we only have to show that each $E(R/\mathfrak{p}_i)$ has the DCC on U-closed submodules. Let then $\mathfrak{p} = \mathfrak{p}_1$ (say). Then $E(R/\mathfrak{p})$ is an injective envelope of the unique simple $R_{\mathfrak{p}}$-module and it follows from [15, Theorem 5] that $E(R/\mathfrak{p})$ has finite length as $R_{\mathfrak{p}}$-module. Thus it will suffice to show that each U-closed submodule of $E(R/\mathfrak{p})$ is, in fact, an $R_{\mathfrak{p}}$-submodule. To see this, let $0 \neq x \in E(R/\mathfrak{p})$. Then $R_{\mathfrak{p}}x$ is an $R_{\mathfrak{p}}$-module of finite length and so we have as in the proof of [20, Theorem 2] that $\mathfrak{p}^k x = 0$ for some $k \geq 1$. Since $\mathrm{Ann}_R(x)$ is a proper U-closed ideal of R, it is clear that there exists a maximal U-closed ideal $\mathfrak{q}$ of R such that $\mathrm{Ann}_R(x) \subseteq \mathfrak{q}$ and, obviously, $\mathfrak{q}$ must be a prime associated to U. Then we get that $\mathfrak{p}^k \subseteq \mathfrak{q}$ and this implies that $\mathfrak{p} \subseteq \mathfrak{q}$. But from [1, 11.26] it follows that all the primes associated to U are minimal among the U-closed prime ideals of R and hence we must have that $\mathfrak{p} = \mathfrak{q}$, so that $\mathfrak{p}$ is the unique prime associated to U which contains $\mathrm{Ann}_R(x)$. Let now $t \in R$ such that $t \notin \mathfrak{p}$. If we denote by $(Rx)^c$ the U-closure of the R-module Rx in $E(R/\mathfrak{p})$, then $t^{-1}x \in (Rx)^c$ if and only if the ideal $(Rx : t^{-1}x)$ belongs to the Gabriel filter of R defined by U [1, p. 50], i.e., if and only if $(Rx : t^{-1}x) \not\subseteq \mathfrak{p}_i$ for $i = 1, \ldots, n$. But, clearly, we have that $Rt+\mathrm{Ann}_R(x)\subseteq(Rx:t^{-1}x)$ and this implies that the condition indeed holds. Thus $(Rx)^c$ is an R_p-module and this ends the proof. □

Remarks. This result shows that the rings considered in [6, Corollary 6.4; 1, Proposition 2.10; 8, Corollary 1.5] have a right Morita duality. A related result is [9, Theorem 3].

We remark that, in the hypotheses of Theorem 9 (conditions (i)–(iv)), ${}_BU_T$ does not always induce a Morita duality, for U_T is not necessarily injective. In [16] a finitely generated projective module ${}_RP$ is constructed over a quasi-Frobenius ring R, such that $T=\mathrm{End}({}_RP)$ is not QF. This module is clearly Δ-injective but is not injective over T.

As we have already remarked, the duality R-modules ${}_RU$ of [23] have the property that U_T is an injective cogenerator but the converse is not true. We now characterize the quasi-injective modules ${}_RU$ such that U_T is an injective cogenerator.

Theorem 10. *Let ${}_RU$ be a quasi-injective module and $T=\mathrm{End}({}_RU)$. Then the following conditions are equivalent:*

(i) *U_T is an injective cogenerator.*

(ii) *${}_RU$ satisfies the following conditions:*

(a) *${}_RU$ cogenerates all the cokernels of homomorphisms of the form $U\to U^n$, i.e. U_T is semi-injective.*

(b) *${}_RU$ is U-linearly compact.*

(c) *The lattice of U-closed submodules of ${}_RU$ has the finite intersection property.*

(iii) *Every cyclic right T-module and every U-cogenerated quotient of U are U-reflexive.*

Proof. The equivalence of conditions (i) and (ii) can be deduced from Theorem 2, together with [7, Theorem 1.5]. However, we give a cyclic proof of the theorem which emphasizes the duality aspects and is independent from [7].

(i) ⇒ (ii) Conditions (ii(a)) and (ii(b)) follow from (i) using Theorem 2. On the other hand, it is clear that, since U_T is a cogenerator, all the right ideals of T are U-closed. They form a lattice which is anti-isomorphic to the lattice of U-closed submodules of U [1, Proposition 3.3] and hence it is clear that this lattice has the finite intersection property.

(ii) ⇒ (iii) First we show that each right ideal J of T is U-closed, that is, that $J=\imath_T\ell_U(J)$. Let $f\in\imath_T\ell_U(J)$, $X=U/\ell_U(J)$ and $p:U\to X$ the canonical projection. Then f factors in the form $f=g\circ p$, with $g:X\to U$. Consider the inverse system $\{X\to X_i\}_I$, where for each finitely generated right ideal $J_i\subseteq J$ we define $K_i=\ell_U(J_i)$ and $X_i=U/K_i$. Since $J=\sum_I J_i$ we have that $\ell_U(J)=\ell_U(\sum_I J_i)=\bigcap_I \ell_U(J_i)=\bigcap_I K_i$ and, since U is U-linearly compact we get from Proposition 1 that $\varprojlim X_i=U/\ell_U(J)=X$. Let now $A_i=\mathrm{Ker}(X\to X_i)$ and $V_i=g(A_i)$. We have a commutative diagram with exact rows and columns,

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} g \cap A_i & \longrightarrow & A_i & \longrightarrow & V_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} g & \longrightarrow & X & \xrightarrow{g} & \operatorname{Im} g & \longrightarrow & 0 \\
 & & {\scriptstyle q_i}\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_i & \longrightarrow & X_i & \longrightarrow & Z_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Now, each N_i, being a submodule of X_i, is finitely U-cogenerated, that is, each $\operatorname{Ker} g \cap A_i$ is a (finitely) U-closed submodule of $\operatorname{Ker} g$. On the other hand, $\operatorname{Ker} g$ is (finitely) U-closed in X and hence a (finitely) U-closed submodule of U. Since U is U-linearly compact, it is straightforward to see that $\operatorname{Ker} g$ is also U-linearly compact and from Proposition 1 it follows that $\varprojlim q_i$ is an epimorphism. Thus, taking inverse limits and observing that $\bigcap A_i = 0$, we get a commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & \bigcap V_i & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} g & \longrightarrow & X & \longrightarrow & \operatorname{Im} g & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \\
0 & \longrightarrow & \varprojlim N_i & \longrightarrow & \varprojlim X_i & \longrightarrow & \varprojlim Z_i & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

which, using the Ker–Coker lemma, shows that $\bigcap V_i = 0$. On the other hand, if we set $Y_i = X_i \oplus \operatorname{Im} g$, for each $i \in I$, then we have exact sequences $U \to Y_i \to Z_i \to 0$ and, since each Y_i is finitely U-cogenerated, our hypothesis (ii(a)) implies that Z_i is U-cogenerated. Then, the fact that $\operatorname{Im} g$ ($= \operatorname{Im} f$) is U-closed in U (again by (ii(a))) implies that the V_i are closed submodules of U. By (ii(c)) the lattice of U-closed submodules of U has the finite intersection property and since the $\{V_i\}$ are an inverse family of submodules we must have that $V_i = 0$ for some $i \in I$. This implies that g (and hence f) factors through the corresponding X_i and so $f \in r_T \ell_U(J_i)$ (for J_i is U-

closed by [1, Proposition 4.1]). Thus $f \in J$ and this completes the proof that J is U-closed. Now, it is easy to see that this implies that T/J is a U-reflexive right T-module. Furthermore, since by Theorem 2, U_T is injective, it clearly follows that each U-cogenerated quotient of U is U-reflexive.

(iii) $\Rightarrow$ (i) Since each simple right T-module is U-reflexive and hence U-cogenerated, it is enough to prove that U_T is injective. Let J be a right ideal of T with inclusion $j : J \to T$. Since T/J is U-reflexive, J is U-closed and hence $J \cong (U/\ell_U(J))^*$. But $U/\ell_U(J)$ is also U-reflexive by hypothesis, so that we get $J^* \cong U/\ell(J)$ and therefore we have a commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\;j^*\;} & J^* \\ \Big\downarrow{\scriptstyle \cong} & & \Big\uparrow{\scriptstyle \cong} \\ U & \longrightarrow & U/\ell_U(J) \end{array}$$

with the vertical arrows isomorphisms, which shows that j^* is an epimorphism and hence U_T is injective. □

Remarks. The artinian injective modules ${}_RU$ which cogenerate an exact torsion theory of R-**Mod** considered in [4, Theorem 2.8], obviously satisfy all the conditions in (ii) of Theorem 10, so that U_T is an injective cogenerator in this case, as claimed in the remarks following Theorem 2.

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